Methods of Point Estimation. Method of Moments

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Motivation

- Practical situation: we know that random data is drawn from a parametric model (distribution), whose parameters we do not know.

For example, in an election between two candidates, data will be drawn from a Bernoulli($p$) distribution with unknown parameter $p$. Use the data to estimate the value of the parameter $p$, as $p$ predicts the result of the election.

- Before... If the distribution (model) and its parameters are known then we can calculate the probability of data.

- Now with Stat. Inference... Estimate the probability of parameters given a parametric model and observed data drawn from it.

I know that data follows the Normal distributions but don’t know the values of the parameters $\mu$ and $\sigma^2$. However, data from a random sample is available to draw inference about $\mu$ and $\sigma^2$. 
Methods of Point Estimation

- How to estimate a parameter?
- Estimating a parameter with its sample analogue is usually reasonable
- Still need a more methodical way of estimating parameters
- Method of Moments (MOM) is the oldest method of finding point estimators
- MOM is simple and often doesn't give best estimates
- Method of maximum likelihood (ML or MLE)
- MLEs have better efficiency properties than MOM estimates. But moment estimators are sometimes easier to compute
- Both ML and MOM can produce unbiased point estimators
- Bayesian Estimation of Parameters: prior information + sample results
Method of Moments

Idea: equate the first $k$ population moments, which are defined in terms of expected values, to the corresponding $k$ sample moments. Solve the system of equations.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the probability distribution (discrete or continuous). The $k$th population moment (or distribution moment) is $E(X^k)$, $k = 1, 2, \ldots$ The corresponding $k$th sample moment is

$$\frac{1}{n} \sum_{i=1}^{n} X_i^k, \quad k = 1, 2, \ldots$$

Example: the first population moment is $E(X) = \mu$, and the first sample moment ($k = 1$) is $\bar{X}$. Thus, by equating the population and sample moments, we find that $\hat{\mu} = \bar{X}$. The sample mean is the moment estimator of the population mean.
Exponential Distribution Moment Estimator

Let $X_1, X_2, \ldots, X_n$ be a random sample from the $Exponential(\lambda)$ distribution. The question: which exponential distribution?!

- need to estimate one parameter $\lambda$, so $k = 1$
- MOM: equate $E(X) = \bar{X}$ (population mean = sample mean)

$$E(X) = 1/\lambda = \bar{X}$$

$$\bar{X} = \frac{1}{\lambda}$$

$$\hat{\lambda} = \frac{1}{\bar{X}}$$

is the moment estimator $\lambda$.

Suppose that the time to failure of an electronic module is exponentially distributed. Eight units are randomly selected and tested, resulting in the following failure time (in hours): 11.96, 5.03, 67.40, 16.07, 31.50, 7.73, 11.10, 22.38. The moment estimate of $\lambda$ is

$$\hat{\lambda} = \frac{1}{\bar{x}} = \frac{1}{21.65} = 0.0462$$
Normal Distribution Moment Estimators

Let $X_1, X_2, ..., X_n$ be a random sample from the $\text{Normal}(\mu, \sigma^2)$ distribution. For the normal distribution, $E(X) = \mu$ and $E(X^2) = \mu^2 + \sigma^2$.

- need to estimate two parameters, so $k = 2$
- MOM: equate

$$
E(X) = \bar{X}, \quad E(X^2) = \frac{1}{n} \sum_{i=1}^{n} X_i^2
$$

$$
\mu = \bar{X}, \quad \mu^2 + \sigma^2 = \frac{1}{n} \sum_{i=1}^{n} X_i^2
$$

Solve

$$
\hat{\mu} = \bar{X}
$$

and

$$
\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} X_i^2 - n(\sum_{i=1}^{n} X_i)^2}{n} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n}
$$

Notice that the moment estimator of $\sigma^2$ is not an unbiased estimator.