Independence. Random variables

Anastasiia Kim

February 5, 2020

Independence

Let A be the event that it rains tomorrow, and suppose that P(A)$=1/3$. Also suppose that I toss a fair coin; let B be the event that it lands heads up, P(B)$=1/2$. What is $P(A|B)$?

Independence

$P(A|B) = P(A) = 1/3$

- The result of my coin toss does not have anything to do with tomorrow's weather
- No matter if B happens or not, the probability of A should not change. This is an example of two independent events
- Two events are independent if one event does not provide any information about the other.

Definition

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

In words, two events are independent if we can obtain the probability of their intersection by multiplying their individual probabilities. Alternatively, A and B are independent if learning that B occurred gives us no information that would change our probabilities for A occurring (and vice versa).

- $P(A|B) = P(A)$ if $P(B) \neq 0$
- n events $A_1, ..., A_n$ are independent if

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2)...P(A_n)$$

$$P(A_1 \cup A_2 \cup ... \cup A_n) = 1 - (1 - P(A_1))(1 - P(A_2))...(1 - P(A_n))$$

- If A and B are independent then
  - A and B^c are independent
  - A^c and B are independent
  - A^c and B^c are independent

Example

Suppose that the probability of being killed in a single flight is $p_c = \frac{1}{4 \cdot 10^6}$ based on available statistics. Assume that different flights are independent. If a businessman takes 25 flights per year, what is the probability that he is killed in a plane crash within the next 20 years?

- The total number of flights that he will take during the next 20 years N $=$ (20)(25)=500
- if $p_s = 1 - p_c$ is the probability that he survives a single flight
- P(he survives N flights) $= p_s^N = (1 - p_c)^N$. The probability that the businessman will be killed in a plane crash within the next 20 years is

$$1 - P(\text{he survives 500 flights}) = 1 - (1 - p_c)^N = 1 - (1 - p_c)^{500} = 12.5 \cdot 10^{-5} \approx \frac{1}{8000}$$

DON'T confuse *independence* and *being disjoint*

Concept	Meaning	Formulas
Disjoint	A and B cannot occur at the same time	$A \cap B = \emptyset$, $P(A \cup B) = P(A) + P(B)$
Independent	A does not give any information about B	$P(A\|B) = P(A), P(B\|A) = P(B)$ $P(A \cap B) = P(A)P(B)$

Consider two events A and B, with $P(A) \neq 0$ and $P(B) \neq 0$. If A and B are disjoint, then they are not independent.

- Proof: since A and B are disjoint, we have

$$P(A \cap B) = 0 \neq P(A)P(B)$$

  Thus, A and B are not independent.

Conditional Independence

Two events A and B are conditionally independent given an event C with $P(C) > 0$ if

$$P(A \cap B|C) = P(A|C)P(B|C)$$

A box contains two coins: a regular coin and one fake two-headed coin (P(H)=1). I choose a coin at random and toss it twice. Define the following events.

- A = {First coin toss results in an H}
- B = {Second coin toss results in an H}
- C = {Coin 1 (regular) has been selected}

Find $P(A|C), P(B|C), P(A \cap B|C), P(A), P(B), P(A \cap B)$. Note that A and B are not independent, but they are conditionally independent given C.

Conditional Independence. Example

A certain baby cries *if and only if* she is hungry, tired, or both. Let C be the event that the baby is crying, H be the event that she is hungry, and T be the event that she is tired. Let $P(C) = c$, $P(H) = h$, and $P(T) = t$, where none of c, h, t are equal to 0 or 1. Let H and T be independent.

- *Find c, in terms of h and t.* Since H and T are independent, we have

$$P(C) = P(H \cup T) = P(H) + P(T) - P(H \cap T) = h + t - ht = c$$

- *Find $P(H|C), P(T|C)$, and $P(H, T|C)$.* By Bayes' rule,

$$P(H|C) = \frac{P(C|H)P(H)}{P(C)} = \frac{h}{c}$$

$$P(T|C) = \frac{P(C|T)P(T)}{P(C)} = \frac{t}{c}$$

$$P(H \cap T|C) = \frac{P(C|H \cap T)P(H \cap T)}{P(C)} = \frac{ht}{c}$$

Conditional Independence. Example

A certain baby cries *if and only if* she is hungry, tired, or both. Let C be the event that the baby is crying, H be the event that she is hungry, and T be the event that she is tired. Let $P(C) = c$, $P(H) = h$, and $P(T) = t$, where none of c, h, t are equal to 0 or 1. Let H and T be independent.

- *Are H and T conditionally independent given C?* No, H and T are not conditionally independent given C, since

$$P(H \cap T|C) = \frac{ht}{c} \neq \frac{ht}{c^2} = P(H|C)P(T|C)$$

  We can also see intuitively why they are not conditionally independent given C: if the baby is crying but not hungry, she must be tired.

Random variables

It is easier to deal with a summary variable than with he original probability structure. Examples of random variables are

- the total number of heads (we might not care at all about the actual head-tail sequence that results)
- the sum of two dice
- the length of time a person have to wait at the bus stop for an ART bus
- the average running time of movies in the theater

Random variables. Definition

Given an experiment with sample space S, a random variable (r.v.) is a function from the sample space S to the real numbers $\mathcal{R}$.

A note on notation: Random variables will always be denoted with uppercase letters and the realized values of the variable (or its range) will be denoted by the corresponding lowercase letters. Thus, the random variable X can take the value x, the random variable Y can take the value y, etc.

Probability function

We need to define a new sample space corresponding to the range of the random variables. Given

- a sample space $S = \{s_1, ..., s_n\}$ with a probability function P
- a random variable X with range $\chi = \{x_1, ..., x_m\}$

We define a *probability function* P_X on χ as

$$P_X(X = x_i) = P(\{s_j \in S : X(s_j) = x_i\})$$

It says that we will observe $X = x_i$ if and only if the outcome of the random experiment is an $s_j \in S$ such that $X(s_j) = x_i$. Note that we will write $P(X = x_i)$ rather than $P_X(X = x_i)$

Random variables

- a random variable X assigns a numerical value X(s) to each possible outcome s of the experiment
- the randomness comes from the fact that we have a random experiment (with probabilities described by the probability function P)
- random variables provide numerical summaries of the experiment in question

Example

Consider the experiment of tossing a fair coin three times. Define the r.v. X to be the number of heads obtained in the three tosses. Define the $P(X = x)$.

Example

Three balls are randomly chosen from an urn containing 3 white, 3 red, and 5 black balls. Suppose that we win \$1 for each white ball selected and lose \$1 for each red ball selected. If we let X denote our total winnings from the experiment, then X is a random variable taking on the possible values 0 $\pm 1, \pm 2, \pm 3$ with probabilities $P(X = x)$. Find the probability that we win money. Check that your result is correct.

- For instance, in order for $X = 0$, either all 3 balls selected must be black or 1 ball of each color must be selected.
- Similarly, the event $\{X = 1\}$ occurs either if 1 white and 2 black balls are selected or if 2 white and 1 red is selected.