Conditional probability

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Conditional probability

Conditional probability provides us with a way to reason about the outcome of an experiment, based on partial information

- In a word guessing game, the first letter of the word is a B. What is the likelihood that the second letter is an O?
- How likely is it that a person has a certain disease given that a medical test was negative?

How should we update our beliefs in light of the evidence we observe?

Definition

If A and B are events with $P(B) > 0$, then the conditional probability of A given B, denoted by $P(A|B)$, is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- A is the event whose uncertainty we want to update
- B is the evidence we observe (or want to treat as given)
- P(B) is the prior probability of A ('prior' means before updating based on the evidence)
- $P(A|B)$ the posterior probability of A ('posterior' means after updating based on the evidence)
- Note that there is no such event as $A|B$

Example

Suppose that an urn contains 8 red balls and 4 white balls. We draw 2 balls from the urn without replacement. If we assume that at each draw each ball in the urn is equally likely to be chosen, what is the probability that both balls drawn are red?

- A $=$ {the first ball drawn is red} and B $=$ {the second ball drawn is red}
- Given the first ball selected is red, there are 7 remaining red balls and 4 white balls, so $P(B|A) = 7/11$
- $P(A) = 8/12 = 2/3$ is the probability to draw red ball out of 12 balls
- the desired probability is $P(A \cap B) = P(A)P(B|A) = (2/3)(7/11) = 14/33$

Of course, we could calculate this probability as $\binom{8}{2}/\binom{12}{2} = 14/33$

Consequences

- For any events A and B with positive probabilities

$$P(A \cap B) = P(B)P(A|B) = P(A)P(B|A)$$

- *Multiplication rule*: for any events $A_1, ..., A_n$ with $P(A_1 \cap A_2 \cap ... \cap A_{n-1}) > 0$,

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2)...P(A_n|A_1 \cap ... \cap A_{n-1})$$

- For example,

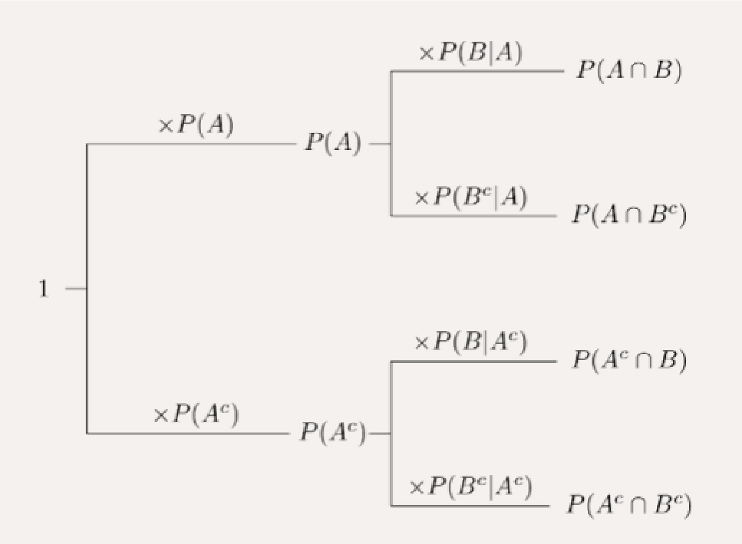
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) = P(A_2)P(A_3|A_2)P(A_1|A_2 \cap A_3)$$

Example

If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

- A $=$ $\{$an aircraft is present$\}$ and B $=$ $\{$the radar generates an alarm$\}$
- Define A^c and B^c
- Find P(not present, false alarm) and P(present, no detection)

Illustration



Bayes' theorem

- Bayes' rule
$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$
- If the odds of an event A are $odds(A) = P(A)/P(A^c)$, then the odds form of Bayes' rule is
$$\frac{P(A|B)}{P(A^c|B)} = \frac{P(A)}{P(A^c)} \cdot \frac{P(B|A)}{P(B|A^c)}$$
  - In words, this says that the posterior odds are equal to the prior odds times the factor $P(B|A)/P(B|A^c)$, which is known in statistics as the likelihood ratio.

The law of total probability

- The law of total probability relates conditional probability to unconditional probability
- Conditional probability can be used to decompose complicated probability problems into simpler pieces, and it is often used in tandem with Bayes' rule

Let $A_1, ..., A_n$ are disjoint events of the sample space S, with $P(A_i) > 0$ for all i. Then for an event B

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

For example, for $i = 2$, the law of total probability is

$$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$$

Example

A certain disease affects about 1 out of 10,000 people. There is a test to check whether the person has the disease. We know that

- the probability that the test result is positive (suggesting the person has the disease), given that the person does not have the disease, is 2%
- the probability that the test result is negative (suggesting the person does not have the disease), given that the person has the disease, is only 1%.

A random person gets tested for the disease and the result comes back positive. What is the probability that the person has the disease?