Hypothesis testing

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May 6, 2020

Large-sample tests on a proportion

Consider a production process that manufactures items that are classified as either acceptable or defective. Modelling the occurrence of defectives with the binomial distribution is usually reasonable when the binomial parameter p represents the proportion of defective items produced.

- The parameter of interest is p, the proportion
- Hypotheses: $H_0 : p = p_0$ and $H_1 : p \neq p_0$
- The test statistic is

$$Z = \frac{X - np_0}{\sqrt{np_0(1 - p_0)}}$$

  where $Z \sim N(0, 1)$
- Use the value of test statistic to reject H_0 if

$$\left| \frac{X - np_0}{\sqrt{np_0(1 - p_0)}} \right| > z_{\alpha/2}$$

Large-sample tests on a proportion. Example

A semiconductor manufacturer produces controllers used in automobile engine applications. The customer requires that the process fallout or fraction defective at a critical manufacturing step not exceed 0.05 and that the manufacturer demonstrate process capability at this level of quality using $\alpha = 0.05$. The semiconductor manufacturer takes a random sample of 200 devices and finds that 4 of them are defective. Can the manufacturer demonstrate process capability for the customer?

- The parameter of interest is p, the proportion of defective items
- Hypotheses: $H_0 : p = 0.05$ and $H_1 : p < 0.05$
- The test statistic is

$$z = \frac{x - np_0}{\sqrt{np_0(1 - p_0)}} = \frac{4 - 200(0.05)}{\sqrt{200(0.05)(1 - 0.05)}} = -1.95$$

- Use the value of test statistic to reject H_0 if

$$|-1.95| > z_\alpha = z_{0.05} = 1.645$$

We reject H_0 and conclude that the process fraction defective p is less than 0.05, the process is capable.

General setting

Let $X \sim Geometric(\theta)$. We observe X and need to decide between

$$H_0 : \theta = \theta_0 = 0.5 \quad H_1 : \theta = \theta_1 = 0.1$$

Design a level 0.05 test ($\alpha = 0.05$).

We choose a threshold $c \in \mathbb{N}$ and compare the observed value of $X = x$ to c. We accept H_0 if $x \leq c$ and reject it if $x > c$. The probability of type I error is given by

$$\begin{aligned} P(\text{type I error}) &= P(\text{Reject } H_0 \mid H_0) \\ &= P(\text{Reject } H_0 \mid \theta = 0.5) \\ &= P(X > c \mid \theta = 0.5) \\ &= \sum_{k=c+1}^{\infty} P(X = k) \quad \left(\text{where } X \sim Geometric(\theta_0 = 0.5)\right) \\ &= \sum_{k=c+1}^{\infty} (1-\theta_0)^{k-1}\theta_0 \\ &= (1-\theta_0)^c \theta_0 \sum_{l=0}^{\infty} (1-\theta_0)^l \\ &= (1-\theta_0)^c. \end{aligned}$$

To have $\alpha = 0.05$, we need to choose c such that $(1-\theta_0)^c \leq \alpha = 0.05$, so we obtain

$$\begin{aligned} c &\geq \frac{\ln \alpha}{\ln(1-\theta_0)} \\ &= \frac{\ln(0.05)}{\ln(.5)} \\ &= 4.32 \end{aligned}$$

Since we would like $c \in \mathbb{N}$, we can let $c = 5$. To summarize, we have the following decision rule: Accept H_0 if the observed value of X is in the set $A = \{1, 2, 3, 4, 5\}$, and reject H_0 otherwise.

General setting

Let $X \sim Geometric(\theta)$. We observe X and need to decide between

$$H_0 : \theta = \theta_0 = 0.5 \quad H_1 : \theta = \theta_1 = 0.1$$

Find the probability of type II error β.

Since the alternative hypothesis H_1 is a simple hypothesis ($\theta = \theta_1$), there is only one value for β,

$$\begin{aligned}\beta &= P(\text{type II error}) = P(\text{accept } H_0 \mid H_1)\\ &= P(X \leq c \mid H_1)\\ &= 1 - (1 - \theta_1)^c\\ &= 1 - (0.9)^5\\ &= 0.41\end{aligned}$$

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal

- Consider tests of hypotheses on the difference in means $\mu_1 - \mu_2$ of two normal distributions with sample sizes n_1 and n_2 where the variances σ_1^2 and σ_2^2 are unknown
- Two possible cases when $\sigma_1^2 = \sigma_2^2 = \sigma^2$ and $\sigma_1^2 \neq \sigma_2^2$
- The normality assumption is required to develop the test procedure, but moderate departures from normality do not adversely affect the procedure.

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal

Case: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \quad H_1 : \mu_1 - \mu_2 \neq \Delta_0$$

Given that the pooled estimator of σ^2, denoted by S_p^2 is

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

Define T that has a t distribution with $n_1 + n_2 - 2$ degrees of freedom as

$$T = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal

Case: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \quad H_1 : \mu_1 - \mu_2 \neq \Delta_0$$

Null hypothesis: $H_0: \mu_1 - \mu_2 = \Delta_0$.

Test statistic:

$$T_0 = \frac{\overline{X}_1 - \overline{X}_2 - \Delta_0}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \tag{10-14}$$

Alternative Hypotheses	*P*-Value	Rejection Criterion for Fixed-Level Tests
$H_1: \mu_1 - \mu_2 \neq \Delta_0$	Probability above $\lvert t_0 \rvert$ and probability below $-\lvert t_0 \rvert$	$t_0 > t_{\alpha/2, n_1+n_2-2}$ or $t_0 < -t_{\alpha/2, n_1+n_2-2}$
$H_1: \mu_1 - \mu_2 > \Delta_0$	Probability above t_0	$t_0 > t_{\alpha, n_1+n_2-2}$
$H_1: \mu_1 - \mu_2 < \Delta_0$	Probability below t_0	$t_0 < -t_{\alpha, n_1+n_2-2}$

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal

Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Specifically, catalyst 1 is currently used; but catalyst 2 is acceptable. Because catalyst 2 is cheaper, it should be adopted, if it does not change the process yield. A test is run in the pilot plant and results in the data shown below. Assume that we have two normal distributions. Is there any difference in the mean yields? Use $\alpha = 0.05$, and assume equal variances.

Observation Number	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
	$\bar{x}_1 = 92.255$	$\bar{x}_2 = 92.733$
	$s_1 = 2.39$	$s_2 = 2.98$

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Equal

Two catalysts are being analyzed to determine how they affect the mean yield of a chemical process. Is there any difference in the mean yields? Use $\alpha = 0.05$, and assume equal variances.

1. **Parameter of interest:** The parameters of interest are μ_1 and μ_2, the mean process yield using catalysts 1 and 2, respectively, and we want to know if $\mu_1 - \mu_2 = 0$.
2. **Null hypothesis:** $H_0: \mu_1 - \mu_2 = 0$, or $H_0: \mu_1 = \mu_2$
3. **Alternative hypothesis:** $H_1: \mu_1 \neq \mu_2$
4. **Test statistic:** The test statistic is

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2 - 0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

5. **Reject H_0 if:** Reject H_0 if the P-value is less than 0.05.
6. **Computations:** From Table 10-1, we have $\bar{x}_1 = 92.255$, $s_1 = 2.39$, $n_1 = 8$, $\bar{x}_2 = 92.733$, $s_2 = 2.98$, and $n_2 = 8$. Therefore

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2} = \frac{(7)(2.39)^2 + 7(2.98)^2}{8 + 8 - 2} = 7.30$$

$$s_p = \sqrt{7.30} = 2.70$$

and

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{2.70\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{92.255 - 92.733}{2.70\sqrt{\frac{1}{8} + \frac{1}{8}}} = -0.35$$

7. **Conclusions:** Because $|t_0| = 0.35$, we find from Appendix Table V that $t_{0.40,14} = 0.258$ and $t_{0.25,14} = 0.692$. Therefore, because $0.258 < 0.35 < 0.692$, we conclude that lower and upper bounds on the P-value are $0.50 < P < 0.80$. Therefore, because the P-value exceeds $\alpha = 0.05$, the null hypothesis cannot be rejected.

Practical Interpretation: At the 0.05 level of significance, we do not have strong evidence to conclude that catalyst 2 results in a mean yield that differs from the mean yield when catalyst 1 is used.

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Not Equal

Case: $\sigma_1^2 \neq \sigma_2^2$

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \quad H_1 : \mu_1 - \mu_2 \neq \Delta_0$$

The statistic T_0^* is distributed approximately as t with degrees of freedom v

$$T = \frac{\bar{X}_1 - \bar{X}_2 - \Delta_0}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}}$$

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1-1} + \frac{(s_2^2/n_2)^2}{n_2-1}}$$

If v is not an integer, round down to the nearest integer.

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Not Equal

Arsenic concentration in public drinking water supplies is a potential health risk. An article in the Arizona Republic reported drinking water arsenic concentrations in parts per billion (ppb) for 10 metropolitan Phoenix communities and 10 communities in rural Arizona. The data (follow normal distribution) follow:

Metro Phoenix ($\bar{x}_1 = 12.5, s_1 = 7.63$)	**Rural Arizona** ($\bar{x}_2 = 27.5, s_2 = 15.3$)
Phoenix, 3	Rimrock, 48
Chandler, 7	Goodyear, 44
Gilbert, 25	New River, 40
Glendale, 10	Apache Junction, 38
Mesa, 15	Buckeye, 33
Paradise Valley, 6	Nogales, 21
Peoria, 12	Black Canyon City, 20
Scottsdale, 25	Sedona, 12
Tempe, 15	Payson, 1
Sun City, 7	Casa Grande, 18

Tests on the Difference in Means of Two Normal Distributions, Variances Unknown and Not Equal

1. **Parameter of interest:** The parameters of interest are the mean arsenic concentrations for the two geographic regions, say, μ_1 and μ_2, and we are interested in determining whether $\mu_1 - \mu_2 = 0$.
2. **Null hypothesis:** $H_0: \mu_1 - \mu_2 = 0$, or $H_0: \mu_1 - \mu_2$
3. **Alternative hypothesis:** $H_1: \mu_1 \neq \mu_2$
4. **Test statistic:** The test statistic is

$$t_0^* = \frac{\bar{x}_1 - \bar{x}_2 - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

5. **Reject H_0 if:** The degrees of freedom on t_0^* are found from Equation 10-16 as

$$v = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_1)^2}{n_2 - 1}} = \frac{\left[\frac{(7.63)^2}{10} + \frac{(15.3)^2}{10}\right]^2}{\frac{\left[(7.63)^2/10\right]^2}{9} + \frac{\left[(15.3)^2/10\right]^2}{9}} = 13.2 \simeq 13$$

Therefore, using $\alpha = 0.05$ and a fixed-significance-level test, we would reject $H_0: \mu_1 = \mu_2$ if $t_0^* > t_{0.025,13} = 2.160$ or if $t_0^* < -t_{0.025,13} = -2.160$.

6. **Computations:** Using the sample data, we find

$$t_0^* = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{12.5 - 27.5}{\sqrt{\frac{(7.63)^2}{10} + \frac{(15.3)^2}{10}}} = -2.77$$

7. **Conclusion:** Because $t_0^* = -2.77 < t_{0.025,13} = -2.160$, we reject the null hypothesis.

Practical Interpretation: There is strong evidence to conclude that mean arsenic concentration in the drinking water in rural Arizona is different from the mean arsenic concentration in metropolitan Phoenix drinking water. Furthermore, the mean arsenic concentration is higher in rural Arizona communities. The P-value for this test is approximately $P = 0.016$.