Counting

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General Information (updated)

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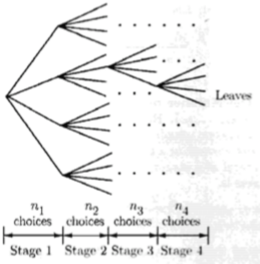
Multiplication rule (fundamental theorem of counting)

If an experiment can be described as a sequence of k stages, and

- the number of ways of completing stage 1 is n_1, and
- the number of ways of completing stage 2 is n_2 for each way of completing stage 1, and
- the number of ways completing stage 3 is n_3 for each way of completing stage 2, and so forth.

The total number of possible results of the k-stage experiment is

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k$$



Example 1

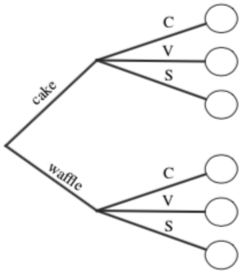


Figure 1: Tree diagram for choosing an ice cream cone. You can choose whether to have a cake cone or a waffle cone, and whether to have chocolate, vanilla, or strawberry as your flavor.

Example 2

A local telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1. How distinct telephone numbers are there?

- Visualize the choice of a subset as a sequential process: select one digit at a time

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A local telephone number is a 7-digit sequence, but the first digit has to be different from 0 or 1. How distinct telephone numbers are there?

- Visualize the choice of a subset as a sequential process: select one digit at a time
- There are 7 stages and we can choose one out of 10 elements at each stage, except for the first stage
- We have $10 - 2 = 8$ choices for the first stage

Using the multiplication rule, the answer is $8 \cdot 10 \cdot 10 \ldots \cdot 10 = 8 \cdot 10^6$.

Example 3

- How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers? 175,760,000
- How many license plates would be possible if repetition among letters or numbers were prohibited? 78,624,000

Example 4

Roll a die 3 times. What is the probability that you get *different* numbers?

- Identify the set of equally likely outcomes
- Compute the total number of outcomes and the number of good outcomes
- Compute the probability as #of good outcomes/total # of outcomes

Permutations

A *permutation* of the elements is an ordered sequence of the elements. The number of permutations of n different elements is

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

- For example, there are $n!$ ways in which n people can line up for ice cream.
- How many different ordered arrangements of the letters a, b, and c are possible?
- 3! (3 letters). By direct enumeration we see that there are 6: abc, acb, bac, bca, cab, and cba. Each arrangement is a permutation. There are 6 possible permutations of a set of 3 objects.

Permutations of subsets

The number of ordered sequences (permutations) of subsets of r elements selected from a set of n different elements is

$$p_r^n = n \times (n-1) \times (n-2) \times ... \times (n-r+1) = \frac{n!}{(n-r)!}$$

If $r = n$, there are $n!$ permutations.

Exercise: count the the number of words that consist of four distinct letters. Answer: 358,800.

Permutations of similar objects. Multinomial coefficient

The number of permutations of $n = n_1 + n_2 + ... + n_r$ elements selected of which n_1 are of one type, n_2 are of a second type, ... , and n_r are of an rth type is

$$\frac{n!}{n_1!n_2!n_3!...n_r!}$$

- A hospital operating room needs to schedule 3 knee surgeries and 2 hip surgeries in a day. What is the number of possible sequences of these surgeries?
- How many different letter arrangements can be formed using the letters *STATISTICS*?

Ordered sampling with replacement

- Select an item at random r times from a collection of n distinct items, replacing the selected item each time before the next selection.
- The total number of ways of choosing r items from a set of n items when ordering matters and repetition is allowed

$$n \times n \times \ldots \times n = n^r$$

- Imagine a jar with 3 balls, labeled from 1 to 3. Sample balls one at a time with replacement, meaning that each time a ball is chosen, it is returned to the jar.
- Each sampled ball is a sub-experiment (stage) with 3 possible outcomes, and there are 2 sub-experiments. Thus, by the multiplication rule there are $3^2 = 9$ ways to obtain a sample of size 2.

Combinations: unordered sampling without replacement

A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

- This is the same as the problem of counting the number of k-element subsets of a given n-element set.
- But forming a combination is different than forming a k-permutation in a combination there is *no ordering* of the selected elements.

For example, whereas the 2-permutations of the letters A, B, and C are
AB, BA, AC, CA, BC, CB
the combinations (no duplicates!) of two out of these three letters are
AB, AC, BC

Combinations

- $n \times (n-1) \times (n-2) \times ... \times (n-r+1) = \frac{n!}{(n-r)!}$ represents the number of different ways that a group of r items could be selected from n items when the order of selection is relevant
- each group of r items will be counted $r!$ times in this count, it follows that the number of different groups of r items that could be formed from a set of n items is $\binom{n}{r}$

$$\binom{n}{r} = \frac{n(n-1)...(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$$

We call $C_r^n = \binom{n}{r}$ *n choose r* or *binomial coefficient*

Exercise: A committee of 3 is to be formed from a group of 20 people. How many different committees are possible?

Example

A bin of 30 parts contains 3 defective parts and 27 nondefective parts. What is the probability of getting exactly 2 defective parts in a sample of size 5 if the sampling is done without replacement (repetition not allowed)?

- Calculate the number of ways we can choose 2 defective parts from the 3 defective parts
- Calculate the number of ways to select the remaining $5 - 2 = 3$ nondefective parts
- Calculate the total number of different subsets of size 5
- Using the multiplication rule, calculate the probability

Number of possible arrangements of size r from n objects

Table 1.2.1. *Number of possible arrangements of size r from n objects*

	Without replacement	With replacement
Ordered	$\frac{n!}{(n-r)!}$	n^r
Unordered	$\binom{n}{r}$	$\binom{n+r-1}{r}$