Continuous Joint Distributions. Covariance and correlation

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Joint Probability Density Function

A joint probability density function $f(x, y)$ for the continuous random variables X and Y for any region R of 2D space is

$$P((X, Y) \in R) = \int \int_R f(x, y) dxdy$$

- $f(x, y) \geq 0$ for all x,y and $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dxdy = 1$
- Marginal pdfs

$$f(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

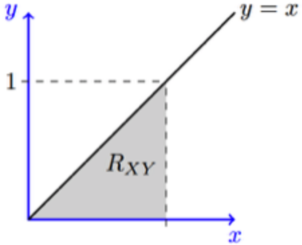
Example

Figure 1: Joint pmf (discrete case) and Joint pdf (continuous case)

Joint Probability Density Function. Example

Given that $f(x, y) = cx^2 y$ for $0 \leq y \leq x \leq 1$ and 0 otherwise,

- the pdf is defined in the region R_{XY}



- constant $c = 10$ since

$$1 = \int_0^1 \int_0^x cx^2 ydydx = \int_0^1 cx^4/2dx = c/10$$

- the marginal pdf of Y is

$$f(y) = \int_y^1 10x^2 ydx = 10y(1 - y^3)/3$$

Continuous form of Bayes' rule and the Law of total probability

For continuous r.v.s X and Y,

- The conditional probability density function of Y given X = x is (for $f(x) > 0$):

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)}$$

$$\int f_{Y|X}(y|x)dy = 1$$

- The continuous form of Bayes' rule is (for $f(x) > 0$):

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f(y)}{f(x)}$$

- The continuous form of the Law of Total Probability is:

$$f(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)f(y)dy$$

Independence of continuous r.v.s

Random variables X and Y are independent if for all x and y,

$$F(x, y) = F(x)F(y)$$

If X and Y are continuous with joint pdf this is equivalent to the condition

$$f(x, y) = f(x)f(y)$$

for all x, y, and it is also equivalent to the condition (for $f(x) > 0$)

$$f_{Y|X}(y|x) = f(y)$$

Example. Exponentials of different rates

Let T_1 be the lifetime of a refrigerator and T_2 be the lifetime of a stove. Let $T_1 \sim \mathit{Exponential}(\lambda_1)$ and $T_2 \sim \mathit{Exponential}(\lambda_2)$ be independent. Find $P(T_1 < T_2)$ the probability that the refrigerator fails before the stove.

Need to integrate the joint pdf of T_1 and T_2 over the appropriate region:

$$P(T_1 < T_2) = \int_0^\infty \int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} dt_1 dt_2 =$$

$$\int_0^\infty \left(\int_0^{t_2} \lambda_1 e^{-\lambda_1 t_1} dt_1 \right) \lambda_2 e^{-\lambda_2 t_2} dt_2 = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Since λ_1 and λ_2 are rates, the probability is 0.5 if $\lambda_1 = \lambda_2$. If, for example, refrigerators have twice the failure rate of stoves $\lambda_1 = 2\lambda_2$, then it says that the odds are 2 to 1 in favor of the refrigerator failing first.

Example. Unit circle

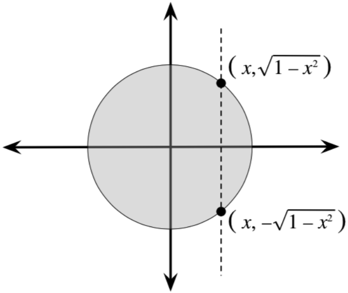
Let (X,Y) be a random point in the unit disk $(x, y) : x^2 + y^2 \leq 1$ with joint pdf

$$f(x, y) = \frac{1}{\pi}, \quad \text{if } x^2 + y^2 \leq 1$$

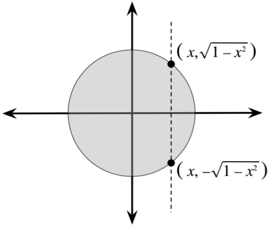
and 0 otherwise.

- Note that X and Y are NOT independent

$$f(0.9, 0.9) = 0 \neq f(x = 0.9)f(y = 0.9)$$



Example. Unit circle



- The marginal distribution of X is

$$f(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2}, \quad -1 \leq x \leq 1$$

- By symmetry, $f(y) = \frac{2}{\pi}\sqrt{1-y^2}$. The conditional distribution of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)f(y)}{f(x)} = \frac{1}{2\sqrt{1-x^2}}, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$$

The conditional pdf is not free of x: X and Y are not independent.

Covariance and Correlation

- mean and variance provided single-number summaries of the distribution of a single r.v.
- covariance is a single-number summary of the joint distribution of two r.v.s.
- covariance measures a tendency of two r.v.s to go up or down together, relative to their means
- positive covariance between X and Y indicates that when X goes up, Y also tends to go up
- negative covariance indicates that when X goes up, Y tends to go down

Covariance

- The covariance between r.v.s X and Y is

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y)$$

- for discrete r.v.s

$$E(XY) = \sum \sum xyp(x,y)dxdy$$

- for continuous r.v.s

$$E(XY) = \int \int xyf(x,y)dxdy$$

- Here $g(x,y) = xy$, $E(g(X,Y)) = E(XY)$ can be thought of as the weighted average of $g(x,y)$ for each point in the range of (X,Y)
- The value of $E(g(X,Y))$ represents the average value of $g(x,y)$ that is expected in a long sequence of repeated trials of the random experiment.

Covariance and Correlation

- if X and Y are independent, then their covariance is zero
- we say that random variables with zero covariance are uncorrelated
- if X and Y are uncorrelated they are not necessary independent
  Let $X \sim N(0,1)$ and let $Y = X^2$. Then $E(XY) = E(X^3) = 0$ because the odd moments of the standard Normal distribution are equal to 0 by symmetry. Thus X and Y are uncorrelated because

  $$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = E(XY) - E(X)E(Y) = 0 - 0 = 0$$

  but they are certainly NOT independent: Y is a function of X, so knowing X gives us perfect information about Y.
- covariance is a measure of linear association, so r.v.s can be dependent in nonlinear ways and still have zero covariance

Correlation

The correlation between r.v.s X and Y is

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

- correlation just scales the covariance by the product of the standard deviation of each variable
- correlation does not depend on the units of measurement
- it is always between -1 and 1
- it is zero for uncorrelated random variables
- the correlation is a measure of the linear relationship between random variables
- if the correlation between two random variables is zero, we cannot immediately conclude that the r.v.s are independent

Covariance and Correlation

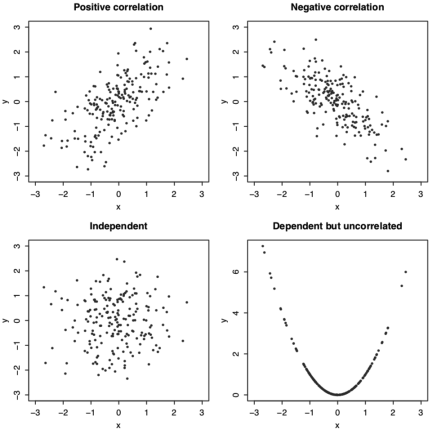


Figure 2: Draws from the joint distribution of (X,Y) under various dependence structures. Bottom left: X and Y are independent, hence uncorrelated. Bottom right: Y is a deterministic function of X ($Y = X^2$), but X and Y are uncorrelated.

Example. Covariance and Correlation

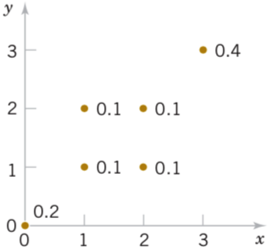


Figure 3: Joint distribution of discrete r.v.s X and Y

$$E(XY) = 0 \cdot 0 \cdot 0.2 + 1 \cdot 1 \cdot 0.1 + 1 \cdot 2 \cdot 0.1 + 2 \cdot 1 \cdot 0.1 + 2 \cdot 2 \cdot 0.1 + 3 \cdot 3 \cdot 0.4 = 4.5$$

$$E(X) = 0 \cdot 0.2 + 1 \cdot 0.2 + 2 \cdot 0.2 + 3 \cdot 0.4 = 1.8$$

$$E(X^2) = 0 \cdot 0.2 + 1 \cdot 0.2 + 4 \cdot 0.2 + 9 \cdot 0.4 = 4.6$$

$$Var(X) = 4.6 - (1.8)^2 = 1.36$$

Example. Covariance and Correlation

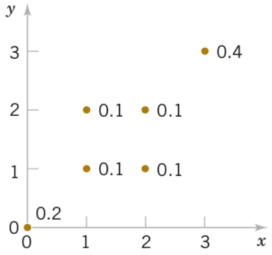


Figure 4: Joint distribution of discrete r.v.s X and Y

The marginal probability distribution of Y is the same as for X, so $E(Y) = 1.8$ and $Var(Y) = 1.36$. Covariance and correlation are

$$Cov(X, Y) = E(XY) - E(X)E(Y) = 4.5 - (1.8)(1.8) = 1.26$$

$$Corr(X, Y) = \frac{1.26}{\sqrt{(1.36)(1.36)}} = 0.93$$