Homework 4. Solution.

Stat 345 - Spring 2020

Problem 1

The maximum time to complete a task in a project is 2.5 days. Suppose that the completion time as a proportion of this maximum is a Beta random variable with mean 0.4 and variance 0.2.

a) Write the probability distribution function (pdf).

$$E(X) = \frac{\alpha}{(\alpha+\beta)}, Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$0.4 = \frac{\alpha}{(\alpha+\beta)}, 0.2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$$\alpha = 0.08, \beta = 0.12$$

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, 0 \leq x \leq 1$$

$$f(x) = \frac{\Gamma(0.2)}{\Gamma(0.08)\Gamma(0.12)} x^{-0.92}(1-x)^{-0.88}, 0 \leq x \leq 1$$

b) What is the probability that the task requires more than two days to complete?

The range for the Beta r.v. is $[0, 1]$. Since $2/2.5 = 0.8$ (the maximum time is 2.5 days),

$$P(\text{more than 2 days}) = P(X > 0.8) = \int_{0.8}^{1} f(x)dx = \frac{\Gamma(0.2)}{\Gamma(0.08)\Gamma(0.12)} \int_{0.8}^{1} x^{-0.92}(1-x)^{-0.88}dx = 0.34$$

Problem 2

The number of cracks in a section of interstate highway that are significant enough to require repair is assumed to follow a Poisson process with rate $\lambda = 2$ cracks per one hour drive. What is the probability that I have to drive at least 3 hours before seeing the first crack?

Since the rate of the Poisson process is 2, the waiting time until the first event is $X \sim Exponential(\lambda = 2)$. The probability that I have to drive at least 3 hours before seeing the first crack is

$$P(X \geq 3) = 1 - F(3) = e^{-3\lambda} = e^{(3)(2)} = e^{-6}$$

Problem 3

The survival function is $S(t) = 1 - F(t)$, or the probability that a person/machine/business lasts longer than t time units. The hazard function is $h(t) = f(t)/S(t)$. Here $F(t)$ is the cdf and $f(t)$ is the pdf. It is the probability that the person/machine/business dies in the next instant, given that it survived to time t. Determine the hazard function for the Exponential(λ) distribution. How does the expression for the exponential hazard function related to the memoryless property of the exponential distribution (explain in words)?

The hazard function for the Exponential(λ) distribution is

$$h(t) = \frac{f(t)}{S(t)} = \frac{f(t)}{1 - F(t)} = \frac{\lambda e^{-\lambda t}}{1 - (1 - e^{-\lambda t})} = \lambda$$

The hazard function (the chances of failure in the next short time interval given that failure hasn't yet occurred) does not change with t.

The exponential distribution is often used to model the failure behavior of electronic components and technical systems because it is assumed that the failures occur randomly during their life span. For example, a 1-month old bulb has the same probability of burning out in the next week as does a 6-month old bulb (the failure probability does not change with its age). Observing that the hazard function is constant is another way of seeing the memoryless property of the exponential distribution.

Problem 4

Suppose that light bulbs have a failure time that is exponentially distributed, with mean survival time 3 months.

a) Suppose $n = 5$ bulbs used simultaneously. What is the probability at most one bulb burns out in the first six months? (*Hint:* first find the probability that the bulb will survive six months. Then model the number of burnt out bulbs with a certain discrete distribution.)

Let Y be a random variable representing a survival time of the light bulb. We know that for the exponential distribution $E(Y) = \frac{1}{\lambda}$. Therefore, $E(Y) = 3$ and $\lambda = \frac{1}{3}$.

The probability that the bulb will burn out in the first 6 months is

$$\text{P(the bulb will burn out in 6 months)} = 1 - \text{P(the bulb will last 6 months)} =$$

$$1 - P(Y > 6) = P(Y < 6) = 1 - e^{-\lambda y} = 1 - e^{-(\frac{1}{3})(6)} = 1 - e^{-2}$$

The number of burnt out bulbs X can be modeled with $Binomial(n = 5, p = 1 - e^{-2})$ distribution. Using Binomial pmf,

$$P(X \leq 1) = P(X = 0) + P(X = 1) = \binom{5}{0}(1 - e^{-2})^0 e^{(-2)(5)} + \binom{5}{1}(1 - e^{-2})^1 e^{(-2)(4)} =$$

$$e^{-10} + 5(1 - e^{-2})e^{-8} = 0.0015$$

b) Let Y_1, Y_2, Y_3, Y_4 be the length of time each bulb survives. What is the distribution of $Y_1 + Y_2 + Y_3 + Y_4$?

Each random variable Y_i is exponentially distributed with the same parameter $\lambda = 3$. The sum of n independent and identically distributed (iid) exponential random variables with the same parameter λ follows $Gamma(n, \lambda)$ distribution. The surviving time of the four bulbs follows $Gamma(4, 3)$ distribution.