Chapter ~~18~~ 20 Concepts

Conditions for Inference about a Mean

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Conditions for Inference About a Mean

- **Random:** The data come from a random sample of size *n* from the population of interest or a randomized experiment.
- **Normal:** The population has a Normal distribution. In practice, it is enough that the distribution be symmetric and single-peaked unless the sample is very small.

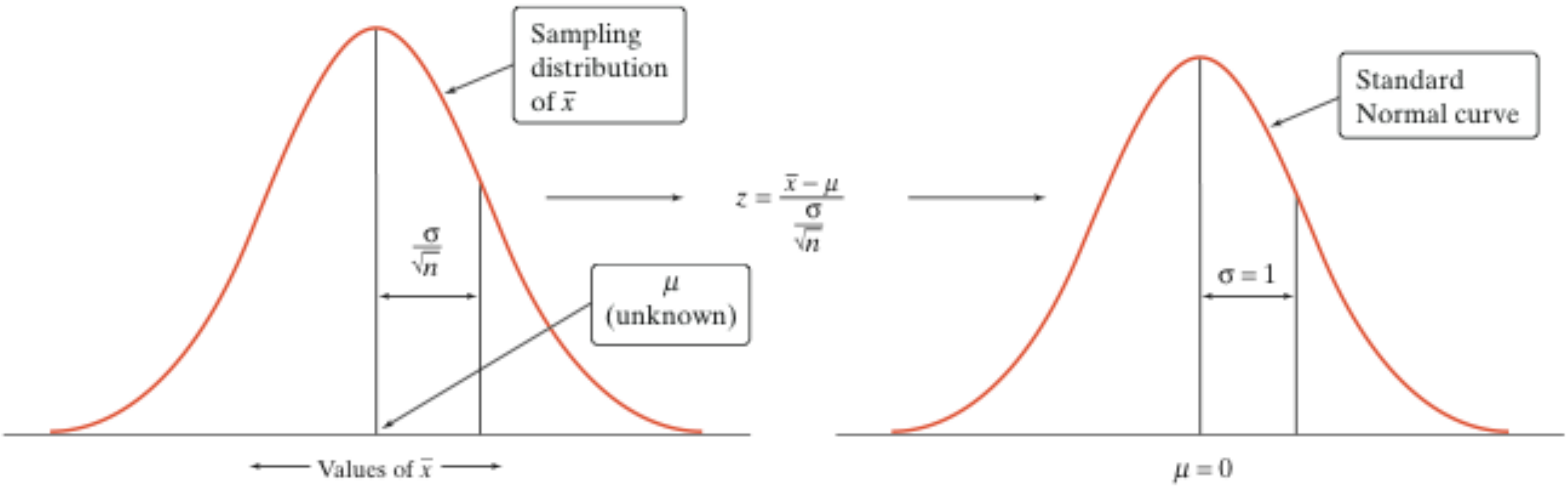
When the conditions for inference are satisfied, the sampling distribution for $\bar{x}$ has roughly a Normal distribution. Because we don't know σ, we estimate it by the sample standard deviation s_x.

The **standard error of the sample mean** $\bar{x}$ is $\frac{s_x}{\sqrt{n}}$, where s_x is the sample standard deviation. It describes how far $\bar{x}$ will be from μ, on average, in repeated SRSs of size n.

The t Distributions

When the sampling distribution of $\bar{x}$ is close to Normal, we can find probabilities involving $\bar{x}$ by standardizing:

$$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$$



When we don't know σ, we can estimate it using the sample standard deviation s_x. What happens when we standardize?

$$?? = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

This new statistic does *not* have a Normal distribution!

The *t* Distributions

When we perform inference about a population mean μ using a t distribution, the appropriate degrees of freedom are found by subtracting 1 from the sample size n, making $df = n - 1$.

The *t* Distributions; Degrees of Freedom

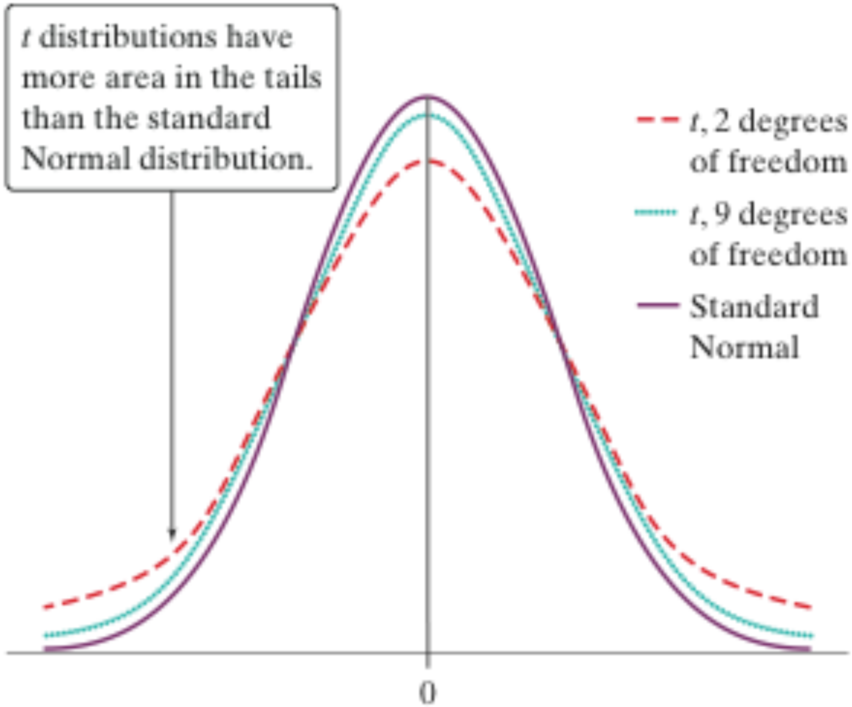
Draw an SRS of size n from a large population that has a Normal distribution with mean μ and standard deviation σ. The **one-sample *t* statistic**

$$t = \frac{\bar{x} - \mu}{s_x / \sqrt{n}}$$

has the ***t* distribution** with **degrees of freedom** $df = n - 1$.

The *t* Distributions

When comparing the density curves of the standard Normal distribution and *t* distributions, several facts are apparent:



- ✓ The density curves of the *t* distributions are similar in shape to the standard Normal curve.
- ✓ The spread of the *t* distributions is a bit greater than that of the standard Normal distribution.
- ✓ The *t* distributions have more probability in the tails and less in the center than does the standard Normal.
- ✓ As the degrees of freedom increase, the *t* density curve approaches the standard Normal curve ever more closely.

We can use Table C in the back of the book to determine critical values t^* for *t* distributions with different degrees of freedom.

Using Table C

Suppose you want to construct a 95% confidence interval for the mean μ of a Normal population based on an SRS of size n = 12. What critical t^* should you use?

	Upper-tail probability *p*			
df	**.05**	**.025**	**.02**	**.01**
10	1.812	2.228	2.359	2.764
11	1.796	2.201	2.328	2.718
12	1.782	2.179	2.303	2.681
z*	1.645	1.960	2.054	2.326
	90%	95%	96%	98%
	Confidence level *C*			

In Table B, we consult the row corresponding to $df = n - 1 = 11$.

We move across that row to the entry that is directly above 95% confidence level.

The desired critical value is $t^* = 2.201$.

One-Sample t Confidence Interval

The **one-sample *t* interval for a population mean** is similar in both reasoning and computational detail to the one-sample *z* interval for a population proportion.

The One-Sample *t* Interval for a Population Mean

Choose an SRS of size n from a population having unknown mean μ. A level C confidence interval for μ is:

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}}$$

where t^* is the critical value for the t_{n-1} distribution.

This interval is exact when the population distribution is Normal and is approximately correct for large n in other cases.

Example

A manufacturer of high-resolution video terminals must control the tension on the mesh of fine wires that lies behind the surface of the viewing screen. The tension is measured by an electrical device with output readings in millivolts (mV). A random sample of 20 screens has the following mean and standard deviation:

$$\bar{x} = 306.32 \text{ mV} \qquad \text{and} \qquad s_x = 36.21 \text{ mV}$$

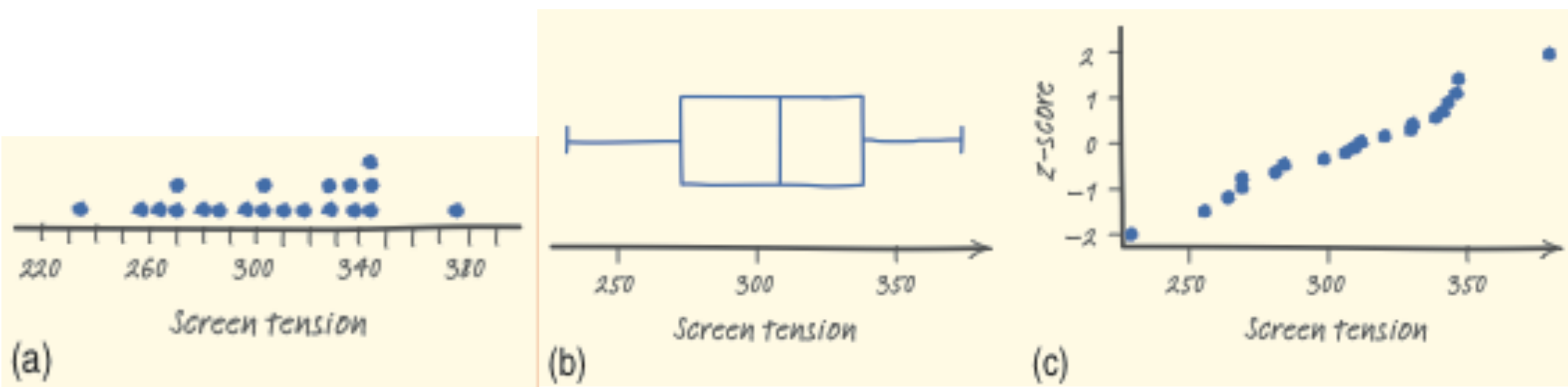
STATE: We want to estimate the true mean tension μ of all the video terminals produced this day at a 95% confidence level.

Example

PLAN: If the conditions are met, we can use a one-sample *t* interval to estimate μ.

Random: We are told that the data come from a random sample of 20 screens from the population of all screens produced that day.

Normal: Since the sample size is small ($n < 30$), we must check whether it's reasonable to believe that the population distribution is Normal. Examine the distribution of the sample data.



These graphs give no reason to doubt the Normality of the population.

Example

DO: We are told that the mean and standard deviation of the 20 screens in the sample are:

$$\bar{x} = 306.32 \text{ mV} \qquad \text{and} \qquad s_x = 36.21 \text{ mV}$$

Upper-tail probability *p*

df	.10	.05	.025
18	1.130	1.734	2.101
19	1.328	1.729	2.093
20	1.325	1.725	2.086
	90%	95%	96%

Confidence level *C*

Since $n = 20$, we use the t distribution with $df = 19$ to find the critical value.

From Table C we find $t^* = 1.729$.

90%

Therefore, the ~~95%~~ confidence interval for μ is:

$$\bar{x} \pm t^* \frac{s_x}{\sqrt{n}} = 306.32 \pm 1.729 \frac{36.21}{\sqrt{20}}$$

$$= 306.32 \pm 14$$

$$= (292.32,\ 320.32)$$

90%

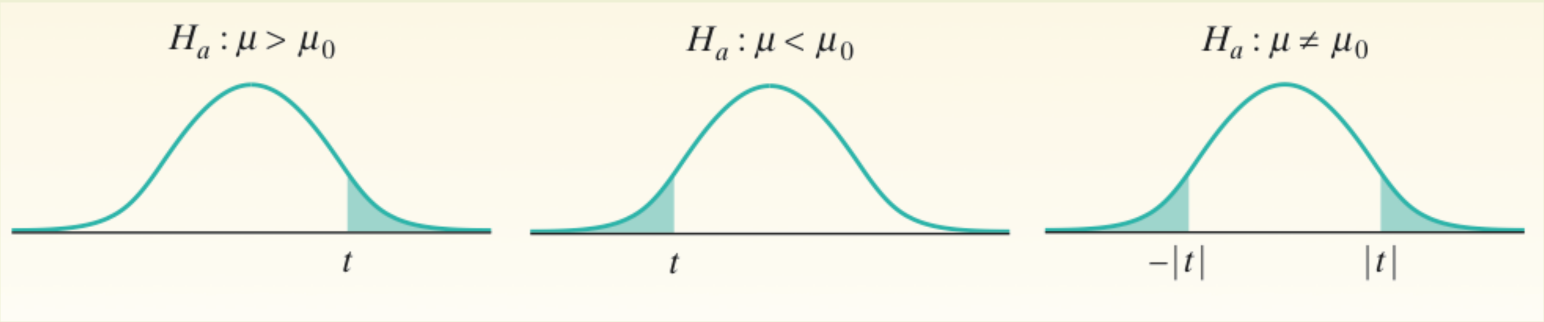
CONCLUDE: We are ~~95%~~ confident that the interval from 292.32 to 320.32 mV captures the true mean tension in the entire batch of video terminals produced that day.

The One-Sample t Test

Choose an SRS of size n from a large population that contains an unknown mean μ. To test the hypothesis $H_0 : \mu = \mu_0$, compute the one-sample t statistic:

$$t = \frac{\bar{x} - \mu_0}{s_x / \sqrt{n}}$$

Find the P-value by calculating the probability of getting a t statistic this large or larger in the direction specified by the alternative hypothesis H_a in a t-distribution with $df = n - 1$.



These P-values are exact if the population distribution is Normal and are approximately correct for large n in other cases.

Example

The level of dissolved oxygen (DO) in a stream or river is an important indicator of the water's ability to support aquatic life. A researcher measures the DO level at 15 randomly chosen locations along a stream. Here are the results in milligrams per liter:

4.53	5.04	3.29	5.23	4.13	5.50	4.83	4.40
5.42	6.38	4.01	4.66	2.87	5.73	5.55	

A dissolved oxygen level below 5 mg/l puts aquatic life at risk.

State: We want to perform a test at the $\alpha = 0.05$ significance level of

$$H_0: \mu = 5$$
$$H_a: \mu < 5$$

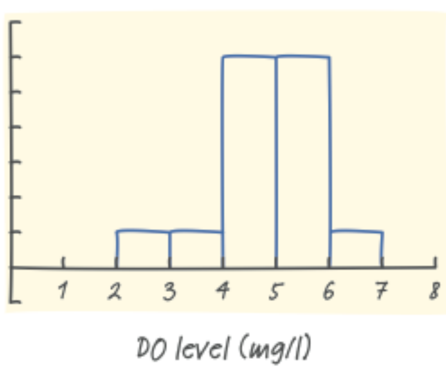
where μ is the actual mean dissolved oxygen level in this stream.

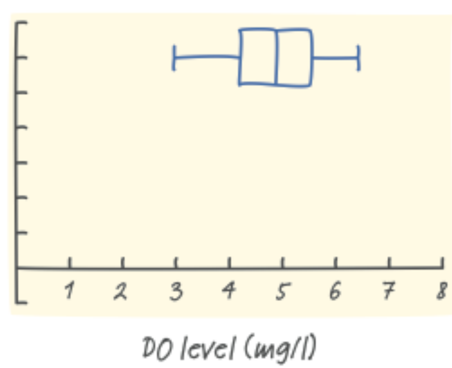
Example

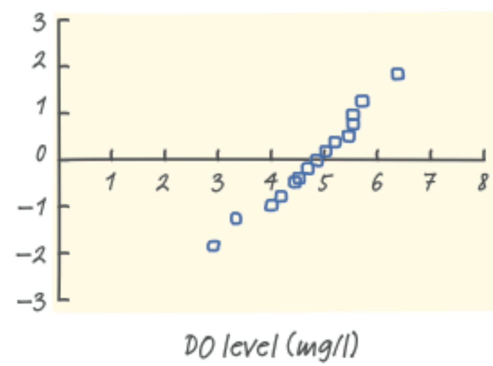
Plan: If conditions are met, we should do a one-sample *t* test for *μ*.

✓ *Random* The researcher measured the DO level at 15 randomly chosen locations.

✓*Normal* We don't know whether the population distribution of DO levels at all points along the stream is Normal. With such a small sample size (*n* = 15), we need to look at the data to see if it's safe to use *t* procedures.





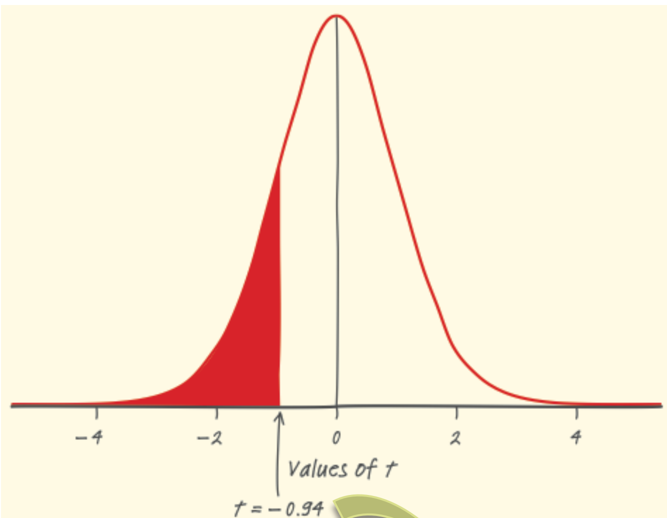


The histogram looks roughly symmetric; the boxplot shows no outliers; and the Normal probability plot is fairly linear. With no outliers or strong skewness, the *t* procedures should be pretty accurate even if the population distribution isn't Normal.

Example

Do: The sample mean and standard deviation are $\bar{x} = 4.771$ and $s_x = 0.9396.$

Test statistic $t = \dfrac{\bar{x} - \mu_0}{s_x / \sqrt{n}} = \dfrac{4.771 - 5}{0.9396 / \sqrt{15}} = -0.94$



***P*-value** The *P*-value is the area to the left of $t = -0.94$ under the t distribution curve with $df = 15 - 1 = 14$.

Upper-tail probability *p*

df	**.25**	**.20**	**.15**
13	.694	.870	1.079
14	.692	.868	1.076
15	.691	.866	1.074
	50%	60%	70%

Confidence level *C*

Conclude: The *P*-value is between 0.15 and 0.20. Since this is greater than our α = 0.05 significance level, we fail to reject H_0. We don't have enough evidence to conclude that the mean DO level in the stream is less than 5 mg/l.

Matched Pairs *t* Procedures

Comparative studies are more convincing than single-sample investigations. For that reason, one-sample inference is less common than comparative inference. Study designs that involve making two observations on the same individual, or one observation on each of two similar individuals, result in **paired data.**

When paired data result from measuring the same quantitative variable twice, as in the job satisfaction study, we can make comparisons by analyzing the differences in each pair. If the conditions for inference are met, we can use one-sample *t* procedures to perform inference about the mean difference μ_d.

Matched Pairs *t* Procedures

To compare the responses to the two treatments in a matched pairs design, find the difference between the responses within each pair. Then apply the one-sample *t* procedures to these differences.

Robustness of *t* Procedures

A confidence interval or significance test is called **robust** if the confidence level or *P*-value does not change very much when the conditions for use of the procedure are violated.

Using the *t* Procedures

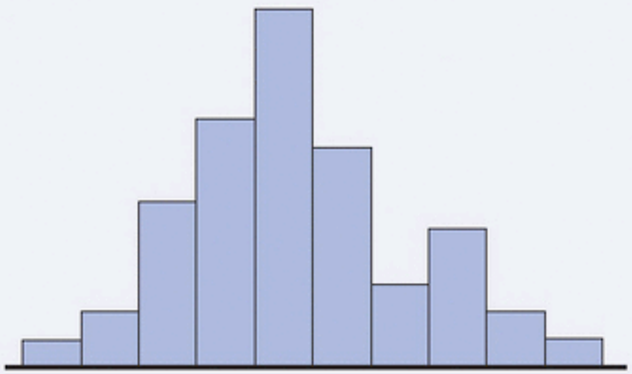
• Except in the case of small samples, the condition that the data are an SRS from the population of interest is more important than the condition that the population distribution is Normal.

• *Sample size less than 15*: Use *t* procedures if the data appear close to Normal. If the data are clearly skewed or if outliers are present, do not use *t*.

• *Sample size at least 15*: The *t* procedures can be used except in the presence of outliers or strong skewness.

• *Large samples*: The *t* procedures can be used even for clearly skewed distributions when the sample is large, roughly $n \geq 40$.

- Figure shows plots of several data sets. For which of these can we safely use the *t* procedures?
- Figure (a) is a **histogram** of the percent of each state's adult residents who are college graduates. *We have data on the entire population of 50 states, so inference is not needed.* We can calculate the exact mean for the population. There is no uncertainty due to having only a sample from the population, and no need for a confidence interval or test. *If these data were an SRS from a larger population*, *t inference would be safe despite the mild skewness because n = 50.*
- Figure (b) is a **stemplot** of the force required to pull apart 20 pieces of Douglas fir. *The data are strongly skewed to the left with possible low outliers, so we cannot trust the t procedures for n = 20.*
- Figure (c) is a **stemplot** of the lengths of 23 specimens of the red variety of the tropical flower *Heliconia. The data are mildly skewed to the right and there are no outliers. We can use the t distributions for such data.*
- Figure (d) is a **histogram** of the heights of the students in a college class. *This distribution is quite symmetric and appears close to Normal. We can use the t procedures for any sample size.*



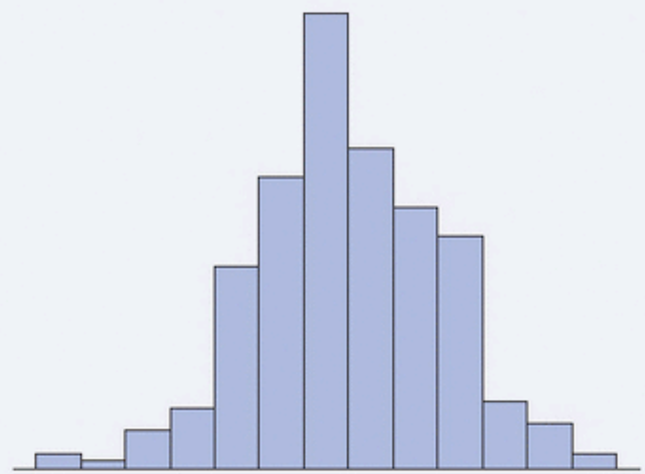
(a)

23	0
24	0
25	
26	5
27	
28	7
29	
30	2 5 9
31	3 9 9
32	0 3 3 6 7 7
33	0 2 3 6

(b)

37	4 8 9
38	0 0 1 1 2 2 8 9
39	2 6 8
40	6 7
41	5 7 9 9
42	0 2
43	1

(c)



(d)

One-Sample *t* Conf. Interval

Which of the following is NOT a true statement about $s/\sqrt{n}$, the standard error of $\bar{x}$?

a) It estimates $\sigma/\sqrt{n}$.

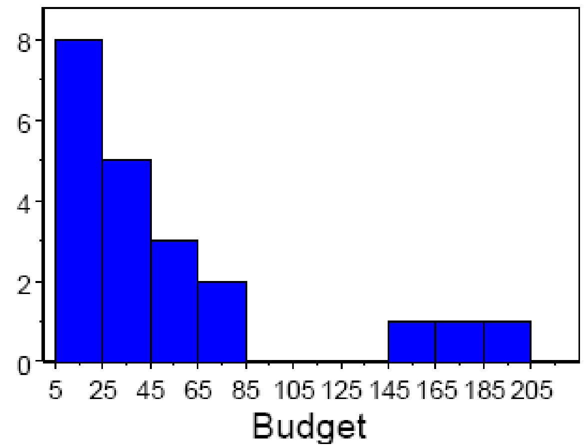
b) We need to know σ when computing it.

c) We use it when we do not know σ.

d) It is an approximation to the standard deviation of $\bar{x}$.

One-Sample *t* Conf. Interval

The following histogram represents the yearly advertising budgets (in millions of dollars) of 21 randomly selected companies. A statistics student wants to create a confidence interval for the mean advertising budget of all companies. By looking at the histogram, is the use of the *t* procedure appropriate in this case?



a) Yes, because data were from an experiment.
b) Yes, because the sample size is large enough.
c) No, because the data are skewed and have outliers.
d) No, because the sampling was not repeated enough times.

One-Sample *t* Conf. Interval

Many people believe that playing chess can have a positive effect on reading ability. A sample of $n = 8$ students was chosen. They participated in a comprehensive chess program, after which they were given a reading test. What is the correct t^* for a 95% confidence interval for the true mean reading score?

a) 1.645
b) 2.306
c) 2.365

TABLE C *t* distribution critical values

d.f.	95%
3	3.182
4	2.776
5	2.571
6	2.447
7	2.365
8	2.306
9	2.262

One-Sample t Conf. Interval

Many people believe that playing chess can have a positive effect on reading ability. A sample of n = 53 students was chosen. They participated in a comprehensive chess program, after which they were given a reading test. What is the correct t^* for a 90% confidence interval for the true mean reading score?

a) 1.645
b) 1.671
c) 1.676
d) 2.009

TABLE C t distribution critical values

d.f.	80%	90%	95%
50	1.299	1.676	2.009
60	1.296	1.671	2.000

One-Sample *t* Test

A train operator claims that her trains are fewer than 7 minutes late on average. A commuter takes an SRS of arriving trains of size n = 21 and records the number of minutes they are late. Here is the stemplot for these data (stem is minutes, leaf is tenths of minutes):

4	9
5	0 4 5 6 9 9
6	0 0 2 3 4 4 6
7	3 5 7
8	0 2

Should the student continue with her use of the *t*-procedure?

a) No, because the sample size is not large enough.
b) Yes, because the data show no strong skewness or outliers.

Matched Pairs *t* Procedures

How do we turn matched pairs data into one-sample data?

a) By taking the maximum within each pair.
b) By taking the differences within each pair.
c) By randomly choosing one value within each pair.
d) By taking the sum for each pair.

Each of 16 milk cows were fed 2 diets in random order. Milk production was measured for each of the cows while being fed each of the 2 diets so that 32 milk production values were recorded. Matched pairs one-sample *t*-procedures will be used for inferences. How many degrees of freedom does the applicable *t*-distribution have?

a) 2
b) 8
c) 15
d) 16
e) 32